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BUCKLING LOADS FOR VARIABLE CROSS-SECTION MEMBERS WITH VARIABLE AXIAL FORCES

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Abstract—This work gives exact solutions for the buckling loads of variable cross-section columns, loaded by variable axial force, for several boundary conditions. Both the cross-section bending stiffness and the axial load can vary along the column as polynomial expressions. The proposed solution is based on a new method that enables one to get the stiffness matrix for the member including the effects of the axial loading. The buckling load is found as the load that makes the determinant of the stiffness matrix equal zero. Several examples are given and compared to published results to demonstrate the accuracy and flexibility of the method. New exact results are given for several other cases.

INTRODUCTION

Columns with non-uniform cross-section are common in engineering. They are used in order to save weight, or to satisfy architectural requirements. Exact buckling loads for some special cases of non-uniform columns were derived in the past. The cases that were treated in the past and solved in this work can be divided into three subsets as follows.

(i) *Variable flexural stiffness with constant axial load*

Gallagher and Lee (1970) gave an approximate finite element solution for monomial variation of the flexural stiffness. Bleich (1952) presented exact solutions for simple monomial stiffness variations. Bert (1984) and Elishakoff and Bert (1988) used improved versions of the Rayleigh method to obtain approximate solutions of variable stiffness columns. Iremonger (1980) solved the problem using the finite difference method. Lately, Smith (1988) gave explicit formulae for the buckling load using the energy method, but these are not satisfactory for design purposes as they have large errors for high taper ratios.

(ii) *Constant flexural stiffness with variable axial load*

Timoshenko and Gere (1961) and Dinnik (1932) presented exact solutions for monomial load variation and simple boundary conditions (cantilever or symmetrically loaded simply supported beam). Frisch-Fay (1966) added the solutions for three more cases of boundary conditions, but only for uniformly distributed axial force.

(iii) *Variable flexural stiffness with variable axial load*

Timoshenko and Gere (1961) and Dinnik (1932) presented exact solutions for monomial variation of both axial and stiffness load for simple boundary conditions. Elishakoff and Pellegrini (1987) presented exact and approximate solutions for two sets of boundary conditions and monomial variation of stiffness and axial load.

When using the finite element method for the case of variable properties of the cross-section along the column, it is common practice to divide it into many small elements, and use some equivalent moment of inertia and axial force for each element. The resulting solutions are approximate, and improvement can be achieved by using larger number of elements. This involves much work in the preparation of data, and results in larger finite element models for solution.

Recently, Eisenberger and Reich (1989b) presented an approximate finite element solution that can be used for variable cross-section members. Later, Eisenberger and Reich

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SAS 27:2-A

(1989a) used this method for all the three subsets that were presented above, and got excellent results for several known cases.

In this paper an exact method for the stability analysis of columns with variable flexural rigidity and variable axial load along their length is introduced. It is based on the derivation of the stiffness matrix for variable cross-section members that was presented in Eisenberger (1989). Here, the stiffness matrix is derived for general polynomial functions of member properties including the effect of the variable axial load. The method is based on the well-known power series solution to differential equations with variable coefficients. However, this is the first time, to the author's knowledge, that this solution has been used to form the stiffness matrix of a variable cross-section member, rather than solve for a particular set of boundary conditions. This stiffness matrix is the exact stiffness matrix, and from that point on the solution for the structure is as for the well-known procedure of the direct stiffness method in matrix analysis of structures. The advantage of having a stiffness matrix is that it can be combined with existing finite element codes directly, and that all the well established procedures that were developed for the finite element method are valid here too. The use of the stiffness method in the solution enables one to treat all combinations of boundary conditions at the same ease (as shown in all the combinations that were solved in the examples), as well as assemblies of members to plane and space frames (with possible application to large structures in space, where variable cross-section members are desirable for weight reduction).

The results of the stability analysis using the proposed method for several examples are compared with results that were obtained using other approximate methods.

STIFFNESS MATRIX CALCULATIONS

The differential equations that govern the bending displacements of a tapered member should be solved in order to obtain the required stiffnesses. The differential equation reads

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right]-\frac{d}{dx}\left[N(x)\frac{dw}{dx}\right]=\frac{d^2}{dx^2}\left[R(x)\frac{d^2w}{dx^2}\right]-\frac{d}{dx}\left[N(x)\frac{dw}{dx}\right]=P(x), \quad (1)$$

where $I(x)$ is the moment of inertia along the beam, w is the lateral displacement, $N(x)$ is the axial force, $P(x)$ is the distributed lateral load along the member, and $R(x) = EI(x)$. The solution for the general case of polynomial variation of $I(x)$, $N(x)$, and $P(x)$ along the beam is not generally available.

Using the finite element technique, it is possible to derive the terms in the stiffness matrix. We assume that the shape functions for the element are polynomials and we have to find the appropriate coefficients. It is widely known that exact terms will result, if one uses the solution of the differential equation as the shape functions, for the derivation of the terms in the stiffness matrix. In this work "exact" shape functions are used, to derive the exact stiffness coefficients. These shape functions are "exact" up to the accuracy of the computer, or up to a preset value set by the analyst.

We take the coefficients in eqn (1) as the following polynomial variation along the beam

$$R(x)=\sum_{i=0}^{j} R_i x^i \quad (2)$$

$$N(x)=\sum_{i=0}^{l} N_i x^i \quad (3)$$

$$P(x)=\sum_{i=0}^{m} P_i x^i, \quad (4)$$

where j, l and m are integers representing the number of terms in each series. This representation is very general, and many functions can be represented in this way, exactly or up to any desired accuracy.

If we introduce a new local variable ξ

$$\xi = \frac{x}{l} \tag{5}$$

we have for eqn (1)

$$\frac{d^2}{d\xi^2}\left[r(\xi)\frac{d^2w}{d\xi^2}\right] - \frac{d}{d\xi}\left[n(\xi)\frac{dw}{d\xi}\right] = p(\xi) \tag{6}$$

with

$$r(\xi) = \sum_{i=0}^{i} R_i L^i \xi^i = \sum_{i=0}^{i} r_i \xi^i \tag{7}$$

$$n(\xi) = \sum_{i=0}^{l} N_i L^{i+2} \xi^i = \sum_{i=0}^{l} n_i \xi^i \tag{8}$$

$$p(\xi) = \sum_{i=0}^{m} P_i L^{i+4} \xi^i = \sum_{i=0}^{m} p_i \xi^i. \tag{9}$$

Now we choose the solution $w(\xi)$ as the following infinite power series

$$w(\xi) = \sum_{i=0}^{L} w_i \xi^i. \tag{10}$$

Calculating all the derivatives and substituting the expressions back into eqn (6) we have

$$
\begin{aligned}
&-\sum_{i=0}^{L}\sum_{k=0}^{i}(k+1)(i-k+1)n_{k+1}w_{i-k+1}\xi^i - \sum_{i=0}^{L}\sum_{k=0}^{i}(i-k+1)(i-k+2)n_k w_{i-k+2}\xi^i \\
&+\sum_{i=0}^{L}\sum_{k=0}^{i}(k+1)(k+2)(i-k+1)(i-k+2)r_{k+2}w_{i-k+2}\xi^i \\
&+\sum_{i=0}^{L}\sum_{k=0}^{i}2(k+1)(i-k+1)(i-k+2)(i-k+3)r_{k+1}w_{i-k+3}\xi^i \\
&+\sum_{i=0}^{L}\sum_{k=0}^{i}(i-k+1)(i-k+2)(i-k+3)(i-k+4)r_k w_{i-k+4}\xi^i = \sum_{i=0}^{n} p_i\xi^i.
\end{aligned} \tag{11}
$$

To satisfy this equation for every value of ξ, we must have

$$
\begin{aligned}
&-\sum_{k=0}^{i}(k+1)(i-k+1)n_{k+1}w_{i-k+1} - \sum_{k=0}^{i}(i-k+1)(i-k+2)n_k w_{i-k+2} \\
&+\sum_{k=0}^{i}(k+1)(k+2)(i-k+1)(i-k+2)r_{k+2}w_{i-k+2} \\
&+\sum_{k=0}^{i}2(k+1)(i-k+1)(i-k+2)(i-k+3)r_{k+1}w_{i-k+3} \\
&+\sum_{k=0}^{i}(i-k+1)(i-k+2)(i-k+3)(i-k+4)r_k w_{i-k+4} = p_i
\end{aligned} \tag{12}
$$

or

$$w_{i+4} = \frac{1}{(i+1)(i+2)(i+3)(i+4)r_0}\left[p_i + \sum_{k=0}^{i}(k+1)(i-k+1)n_{k+1}w_{i-k+1}\right.$$

$$+\sum_{k=0}^{i}(i-k+1)(i-k+2)n_k w_{i-k+2} - \sum_{k=0}^{i}(k+1)(k+2)(i-k+1)(i-k+2)r_{k+2}w_{i-k+2}$$

$$-\sum_{k=0}^{i}2(k+1)(i-k+1)(i-k+2)(i-k+3)r_{k+1}w_{i-k+3}$$

$$\left.-\sum_{k=1}^{i}(i-k+1)(i-k+2)(i-k+3)(i-k+4)r_k w_{i-k+4}\right]. \quad (13)$$

The terms for w_{i+4} tend to 0 as $i \to \infty$. Now we have all the w_i coefficients except for the first four, that should be found using the boundary conditions. For this case we choose as degrees of freedom in the formulation the lateral deflection and rotation at the two ends of the beam element. At $\xi = 0$ we have

$$w_0 = w(0) \quad (14)$$

and

$$w_1 = w'(0) \quad (15)$$

so the first two terms are readily known from the boundary conditions.

The terms w_2 and w_3 are found as follows: All the w_is are linearly dependent on the first four, and we can write

$$w(1) = \sum_{i=0}^{\infty} w_i = C_0w_0 + C_1w_1 + C_2w_2 + C_3w_3 + \sum_{i=0}^{\infty} C_{pi}p_i \quad (16)$$

$$w'(1) = \sum_{i=1}^{\infty} iw_i = C'_0w_0 + C'_1w_1 + C'_2w_2 + C'_3w_3 + \sum_{i=0}^{\infty} C'_{pi}p_i. \quad (17)$$

The 10 C coefficients (C_0, C_1, C_2, C_3, C'_0, C'_1, C'_2, C'_3, C_{pi}, and C'_{pi}) are expressible in terms of all the coefficients in $r(\xi)$, $n(\xi)$ and $p(\xi)$. C_0 for example, is the value of $w(1)$ when $w_0 = 1$ and $w_1 = w_2 = w_3 = p_i = 0$ calculated from eqn (10) using the recurrence formula in eqn (13). In general we can write all the C coefficients as follows:

$$C_i = w(1) = \sum_{k=0}^{\infty} w_k = 1 + \sum_{k=4}^{\infty} w_k \quad (18)$$

$$C'_i = w'(1) = \sum_{k=1}^{\infty} kw_k = i + \sum_{k=4}^{\infty} kw_k \quad (19)$$

both with w_k [from eqn (13)] based on $w_i = 1$, $w_{k \neq i} = p_k = 0$; $i, k = 0, 1, 2, \ldots, \infty$, and

$$\sum_{i=0}^{\infty} C_{pi} = w(1) = \sum_{k=4}^{\infty} w_k \quad (20)$$

$$\sum_{i=0}^{\infty} C'_{pi} = w'(1) = \sum_{k=4}^{\infty} kw_k \quad (21)$$

both with w_k [from eqn (13)] based on $w_i = 0$; $i = 0, 1, 2, 3$, and using the values p_i for the particular loading. Then, knowing all the terms in eqns (18)–(21), the values of w_0 and w_1

[eqns (14)–(15)], and the boundary conditions at $x = L(\xi = 1)$ we can solve eqns (16) and (17) and find the unknowns w_2 and w_3. Thus, for any given variable polynomial functions [eqns (7)–(9)] we can find all the coefficients w_i in eqn (13).

The terms in the stiffness matrix can be found as in the finite element method using the following expression

$$S = \int_0^1 F''^{\mathrm{T}}(\xi) EI(\xi) F''(\xi)\,\mathrm{d}\xi \tag{22}$$

where $F''(\xi)$ are the second derivatives of the basis functions. The four basis functions F (also called shape functions) are found using eqns (10), (14)–(15) and (16)–(17) for an unloaded member [i.e. $p(x) = 0$] with the following boundary conditions:

(1) $w(0) = 1;\ w'(0) = w(1) = w'(1) = 0;$

(2) $w'(0) = 1;\ w(0) = w(1) = w'(1) = 0;$

(3) $w(1) = 1;\ w(0) = w'(0) = w'(1) = 0;$

(4) $w'(1) = 1;\ w(0) = w'(0) = w(1) = 0.$

The shape functions that are found using this technique have the special property that they are the "exact" solution for the differential equation. The word exact in the previous sentence stands for "as exact as we can get on a digital computer". This is so since the calculation of the C coefficients is stopped according to a preset criteria: it could be until the contribution of the next element is less than an arbitrary small ε (in most of the cases ε was chosen as 10^{-18}) or until the C values converge completely (for the accuracy of the computer). In this work, the terms in the stiffness matrix are found in a simpler and faster way using the properties of the shape functions [rather than by eqn (22)], as follows: the terms in the stiffness matrix are defined as the holding actions at both ends of the beam, due to unit translation or rotation, at each of the four degrees of freedom, one at a time. Thus, corresponding to the four sets of boundary conditions above there are four solutions W_i; $i = 1, 2, 3, 4$ for $w(\xi)$ which are found using eqns (10), (13) and (16)–(21).

Then, the holding actions will be:

$$\begin{aligned} V(0) &= \frac{r(0)}{L^3}\frac{\mathrm{d}^3 W_i}{\mathrm{d}\xi^3} + \frac{1}{L^3}\frac{\mathrm{d}r(0)}{\mathrm{d}\xi}\frac{\mathrm{d}^2 W_i}{\mathrm{d}\xi^2} + \frac{n(0)}{L}\frac{\mathrm{d}W_i}{\mathrm{d}\xi} \\ &= 6\frac{r(0)}{L^3}W_{i,3} + 2\frac{r'(0)}{L^3}W_{i,2} + \frac{n(0)}{L}W_{i,1} \end{aligned} \tag{23}$$

$$M(0) = -\frac{r(0)}{L^2}\frac{\mathrm{d}^2 W_i}{\mathrm{d}\xi^2} = -2\frac{r(0)}{L^2}W_{i,2} \tag{24}$$

$$\begin{aligned} V(1) &= -\frac{r(1)}{L^3}\frac{\mathrm{d}^3 W_i}{\mathrm{d}\xi^3} - \frac{1}{L^3}\frac{\mathrm{d}r(1)}{\mathrm{d}\xi}\frac{\mathrm{d}^2 W_i}{\mathrm{d}\xi^2} - \frac{n(1)}{L}\frac{\mathrm{d}W_i}{\mathrm{d}\xi} \\ &= -\frac{r(1)}{L^3}\sum_{k=3}^{\infty} k(k-1)(k-2)W_{i,k} - 2\frac{r'(0)}{L^3}\sum_{k=2}^{\infty} k(k-1)W_{i,k} - \frac{n(1)}{L}\sum_{k=1}^{\infty} kW_{i,k} \end{aligned} \tag{25}$$

$$M(1) = \frac{r(1)}{L^2}\frac{\mathrm{d}^2 W_i}{\mathrm{d}\xi^2} = \frac{r(1)}{L^2}\sum_{2}^{\infty} k(k-1)W_{i,k}, \tag{26}$$

where V is the shear force and M is the moment.

$$S(1,i) = 6\frac{r(0)}{L^3}W_{i,3}+2\frac{r'(0)}{L^3}W_{i,2}+\frac{n(0)}{L}W_{i,1} \quad (27)$$

$$S(2,i) = -2\frac{r(0)}{L^2}W_{i,2} \quad (28)$$

$$S(3,i) = -\frac{r(1)}{L^3}\sum_{k=3}^{\infty}k(k-1)(k-2)W_{i,k}-\frac{r'(1)}{L^3}\sum_{k=2}^{\infty}k(k-1)W_{i,k}-\frac{n(1)}{L}\sum_{k=1}^{\infty}kW_{i,k} \quad (29)$$

$$S(4,i) = \frac{r(1)}{L^2}\sum_{k=2}^{\infty}k(k-1)W_{i,k} \quad (30)$$

where $W_{i,k}$ are calculated using the r_n coefficients.

Then the buckling load for variable cross-section members, or frames with such members, can be found as the axial loads $N(x)$ in the members, that cause the determinant of the corresponding stiffness matrix to become zero. This is done using a routine that converges on the values of the axial load that satisfy this criteria. The procedure was incorporated into a regular beam analysis program and demonstrated in the following examples.

At this point, before going into examples, an overall discussion and comparison of the proposed method with the finite element method is presented: one can look at the procedure suggested in this work as an addition to the finite element method, as one developing a methodology to derive shape functions that yield the exact stiffness matrix. When using the finite element method, one can converge to the solution. However, it will take several solutions with increasing number of elements in order to apply an error estimate that will yield a very good, but still approximate solution. Using the proposed method this is not needed and the exact solution is found from the first analysis. From the computational point of view, it is obvious that it is more time consuming to derive the exact stiffness matrix as outlined in this work. But, when this is viewed in comparison to assembling the stiffness matrix for 20 or 50 elements, and the fact that the size of the eigenvalue problem that results in the stability analysis, is much smaller, more than offsets the longer derivation time. As an example, for a fixed-free column with variable cross-section, and variable axial load (such as own weight), a 20 element finite element model, that results in very good estimate of the buckling load (as shown in an example in the next section), leads to a 40 by 40 eigenvalue problem, compared with a 2 by 2 matrix for the proposed method.

In the examples that follow, the power of the new method is demonstrated in the solution of many cases where exact solutions were not available. Also, some comparisons that were made to the wrong values [(Bert, 1984) compared to the result in (Swenson, 1952)] are pointed out.

EXAMPLES

The method was first checked for the classic Euler buckling cases for columns. For all the cases, the method yielded the exact theoretical solutions, using only one element for the whole member. The examples in this section are divided according to the three cases that were presented in the introduction.

(i) *Variable flexural stiffness with constant axial load*

Consider the column that was solved by Swenson (1952) and later by Bert (1984) and Elishakoff and Bert (1988). The member moment of inertia is given as

$$I(\xi) = I_0(1+\xi). \quad (31)$$

The column is loaded by an end load P. This column was solved previously for the case of

Table 1. Values of $\bar{N}$ in eqn (32) for members with variable flexural stiffness and constant axial loads

Boundary conditions		Buckling load	
Strong end	Weak end	Example 1 Swenson (1952)	Example 2 Bleich (1952)
Free	Fixed	3.117696228	3.836376918
Fixed	Free	4.124184446	6.731865407
Hinged	Hinged	14.511249540	20.792288456
Hinged	Fixed	29.448962806	42.109176122
Fixed	Hinged	29.478844262	42.109176122
Fixed	Fixed	57.393956136	81.923363881

simple supports at both ends (and the reported result was incorrect). The values for the nondimensional buckling load $\bar{N}$

$$\bar{N} = \frac{NL^2}{EI_0} \tag{32}$$

are given in Table 1. Also results are given for five more combinations of boundary conditions. In all the examples only one element was used to find the critical load, except for the fixed–fixed case where two elements were used (but only two degrees of freedom). It should also be noted that all the results that are presented in this paper were checked against the converged values that were obtained using the approximate method in Eisenberger and Reich (1989a).

Another example is the column that was solved by Bleich (1952) and later by Bert (1984) and Elishakoff and Bert (1988). For this example, the moment of inertia along the column varied as

$$I(\xi) = I_0(1+\xi)^2. \tag{33}$$

The values of the normalized buckling load are given in Table 1 for the six combinations of boundary conditions. Bleich solved exactly for the hinged–hinged case and obtained the same value. It should be noted here that for this special member the buckling loads for the fixed–hinged and hinged–fixed case are exactly the same. This is only due to the particular variation in cross-section properties, and for small deviation from it, this no longer holds.

(ii) *Constant flexural stiffness with variable axial load*

The third example is of a column with constant flexural stiffness and distributed load along the member that was solved by Dinnik (1932) and is given by Timoshenko and Gere (1961) on p. 131. The variation of the distributed load along the column is given by:

$$q(x) = q_b\xi^p \tag{34}$$

where the subscript b indicates the values at the base of the column. Then, the critical loads are given as

$$P_{cr} = \frac{1}{L}\int_0^L q_b\xi^p\,\mathrm{d}\xi = \frac{mEI}{L^2}. \tag{35}$$

Table 2. Values of m in eqn (35) for members with constant flexural stiffness and variable axial loads

Boundary conditions		Buckling loads					
Upper end	Lower end	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
Free	Fixed	7.837347	16.100953	27.256905	41.304808	58.244502	78.075911
Hinged	Hinged	18.568725	23.238937	26.674598	29.745281	32.703955	35.630368
Fixed	Hinged	30.009421	36.762826	41.916950	46.347724	50.460719	54.434713
Hinged	Fixed	52.500663	78.982899	104.048055	130.353568	158.895394	190.048453
Fixed	Fixed	74.628569	107.823212	139.541434	171.544095	205.037024	240.672113

Table 3. Values of $\bar{N}$ in eqn (32) for members with variable flexural stiffness and variable axial loads

Boundary conditions		Buckling load					
		Approximate—FE					
Strong end	Weak end	2 elements	5 elements	10 elements	20 elements	50 elements	Exact
Fixed	Free	11.905213	13.557930	13.803883	13.865669	13.884204	13.886289
Hinged	Hinged	29.581692	29.257630	29.211988	29.200522	29.197909	29.196698
Hinged	Fixed	58.427582	47.227473	46.441741	46.257822	46.208650	46.197566
Fixed	Hinged	65.926002	82.337170	83.945711	84.348360	84.466380	84.483161
Fixed	Fixed	119.051369	117.775740	117.610867	117.617918	117.630016	117.626024

Timoshenko and Gere (1961) presented the solution for the free–fixed case. In Table 2 the exact values of m that were calculated using the proposed method are shown, and they agree with the values in Timoshenko and Gere (1961). Table 2 contains also values for the free–fixed case that were not given by Timoshenko and Gere (1961) for $n = 4,5$, and these values agree with the converged approximate values that were reported by Eisenberger and Reich (1989a). Four more cases of boundary conditions combinations are given in Table 2. Frisch-Fay (1966) added the solutions for three more cases of boundary conditions, but only for uniformly distributed axial force. The values that he gave were: 18.53 for the hinged–hinged case; 52.49 for the hinged–fixed case; and 74.65 for the fixed–fixed case. It can be seen that these are still approximations, probably due to the accuracy of the calculation that he performed, as his method is exact. All other cases appear here, apparently, for the first time.

It should be noted that the results for the higher values of p in Table 2 indicate that the type of the restraint in the lower end of the column is more significant in the final result for the buckling load. This is so, as the load is concentrated more in the lower part of the column as p is increased, and the upper half of the column is hardly loaded, so that it's effect is just in restraining the shape at the top.

(iii) *Variable flexural stiffness with variable axial load*

The only results that are available for this case are those given by Timoshenko and Gere (1961). However, in all these cases, the moment of inertia at the top of the column was taken as zero, which is not realistic. Such cases with zero stiffness, cannot be solved using the method presented in this work. Therefore, for this case, the results will be compared to those from the finite element method. When using the finite element method, the member flexural rigidity and the axial are taken as constant all along the element, as the value at the mid length point of the element. As an example, the column in the first example that was solved by Swenson (1952), but with uniformly distributed load along the member, will be used here. The load is taken in such a way that the maximum axial load is at the stronger end. In Table 3 the results are given for five combinations of boundary conditions, and compared with the results from the approximate solution using 2, 5, 10, 20 and 50 elements along the member. It is seen that the approximate results converge to the exact results for all the cases. There are two problems with the well-known finite element solution in these cases: the first is that the relative errors are not known and several runs are needed to find if the solution is within some error criteria. The second is that the convergence is for some cases conservative (i.e. the exact buckling load is below the finite element solution) and in other cases it is nonconservative estimate. Overall, the computer time for the more exact finite element solutions (20 and 50 elements) was longer than the time for the exact solution as presented in this work. There is also the guarantee that only one solution is needed and that it will yield the exact solution, when using the proposed method.

(iv) *Sway buckling of a frame*

Another example is that of the sway buckling of the frame in Fig. 1. The frame is composed of four tapered members with linearly varying moment of inertia, with end values

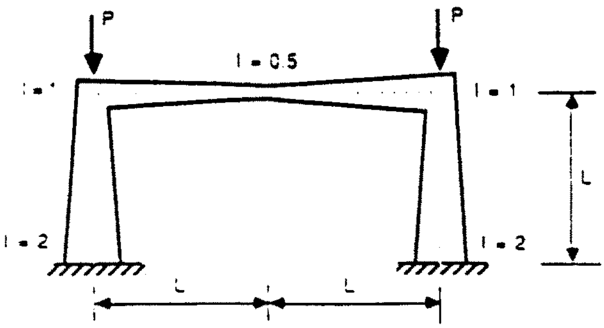

Fig. 1. Example frame with variable cross section members.

as shown. Utilizing symmetry of the problem, only two members were used for the solution in the exact method and the normalized buckling load is 6.017535.

DISCUSSION AND SUMMARY

The method that was presented in this work is based on the solution of the differential equation for any polynomial variation of the cross-section properties. Then, the results for the buckling loads are exact. The application of different sets of boundary conditions is straightforward as in the standard stiffness method of analysis. The first advantage of the method is that it gives exact values for the buckling load (rather than approximate in other methods). Comparing this method to the finite element method or the finite difference method points out the second advantage of the method : only one element is needed for the solution. Thus, the results are computed much faster. The method was used also to find the natural frequencies of vibrations of variable cross-section members (Eisenberger, 1989).

In this work, exact buckling loads (up to the accuracy of the computer) for variable cross-section members with variable axial loads are given. These were derived using a new element based method that enables one to find the stiffness matrix for members with any polynomial variation of the cross-section and axial loading. In the examples, for the three classes that were listed in the introduction, it is shown that the method gives exact results compared to known buckling loads. Many new exact values for buckling loads are given for various combinations of boundary conditions at the ends of the member. This procedure can be incorporated into regular frame programs to yield exact buckling loads for more complex structures.

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